REIDEMEISTER NUMBER OF ANY AUTOMORPHISM OF BAUMSLAG - SOLITAR GROUP IS INFINITE

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Abstract. Let $\phi : G \to G$ be a group endomorphism where G is a finitely generated group of exponential growth, and let $R(\phi)$ denote the number of ϕ-conjugacy classes. Fel'shtyn and Hill [10] conjectured that if ϕ is injective, then $R(\phi)$ is infinite. This conjecture is true for automorphisms of non-elementary Gromov hyperbolic groups, see [28] and [7]. It was shown in [18] that the conjecture does not hold in general. Nevertheless in this paper, we show that the conjecture holds for the Baumslag-Solitar groups $B(m, n)$, where either $|m|$ or $|n|$ is greater than 1 and $|m| \neq |n|$. We also show that in the cases where $|m| = |n| > 1$ or $mn = -1$ the conjecture is true for automorphisms. In addition, we derive few results about the coincidence Reidemeister number.

1. Introduction

J. Nielsen introduced the fixed point classes of a surface homeomorphism in [29]. Subsequently, K. Reidemeister [30] developed the algebraic foundation of the Nielsen fixed point theory for any map of any compact polyhedron. As a result of Reidemeister's work, the twisted conjugacy classes of a group homomorphism were introduced. It turns out that the fixed point classes of a map can easily be identified with the conjugacy classes of lifting of this map to the universal covering of compact polyhedron, and conjugacy classes of lifting can be identified with the twisted conjugacy classes of the

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homomorphism induced on the fundamental group of the polyhedron. Let G be a finitely generated group and let $\phi : G \to G$ be an endomorphism. Two elements $\alpha, \alpha' \in G$ are said to be $\phi-$*conjugate* if there exists $\gamma \in G$ with $\alpha' = \gamma\alpha\phi(\gamma)^{-1}$. The number of ϕ-conjugacy classes(or twisted conjugacy classes) is called the *Reidemeister number* of an endomorphism ϕ, denoted by $R(\phi)$. If ϕ is the identity map, then the ϕ-conjugacy classes are the usual conjugacy classes in the group G. Let X be a connected compact polyhedron and $f : X \to X$ be a continuous map. The Reidemeister number $R(f)$, which is simply the cardinality of the set of ϕ-conjugacy classes where $\phi = f_\#$ is the induced homomorphism on the fundamental group, is relevant for the study of fixed points of f in the presence of the fundamental group. In fact the finiteness of Reidemeister number plays an important rôle. See for example [33], [19], [10], [13] and the introduction of [18].

A current important problem concerns obtaining a twisted analogue of the celebrated Burnside-Frobenius theorem [10, 12, 14, 11, 13]. For this purpose it is important to describe the class of groups G, such that $R(\phi) = \infty$ for any automorphism $\phi : G \to G$. A. Felshtyn and R. Hill [10] made first attempts to localize this class of groups.

Later it was proved in [7, 28] that the non-elementary Gromov hyperbolic groups belong to this class. Furthermore, using the co-Hofian property, it was shown in [7] that, if in addition G is torsion-free and freely indecomposable, then $R(\phi)$ is infinite for every injective endomorphism ϕ. This result gives supportive evidence to a conjecture of [10] which states that $R(\phi) = \infty$ if ϕ is an injective endomorphism of a finitely generated torsion-free group G with exponential growth.

This conjecture was shown to be false in general. In [18] were constructed automorphisms $\phi : G \to G$ on certain finitely generated torsion-free exponential growth groups G that are not Gromov hyperbolic with $R(\phi) < \infty$.

In the present paper we study this problem for a family of finitely generated groups which have exponential growth but are not Gromov hyperbolic. These are the Baumslag-Solitar groups, which we now define. Being indexed by pairs of integer numbers different from zero, they have the following presentation:

$$B(m, n) = \langle a, b : a^{-1}b^m a = b^n\rangle, m, n \neq 0.$$

The present work extends substantially in several directions the preliminary results obtained in [8], and simplifies some of the proofs.

The family of the Baumslag-Solitar groups has different features from the one given in [18], which is a family of groups which are metabelian having as the kernel the group $\mathbb{Z}^n$. Hence they contain a subgroup isomorphic to $\mathbb{Z}+\mathbb{Z}$. In the case of Baumslag-Solitar groups this happens if, and only if, $m = n$. For $m = n = 1$ the group $B(1,1) = \mathbb{Z}+\mathbb{Z}$ does not have exponential growth and it is also known that there are automorphisms $\phi : B(1,1) \to B(1,1)$ with $R(\phi) < \infty$. For more details about these groups $B(m,n)$ see [1, 5].

Some results in this work could be obtained by means of the classification of some of the endomorphisms of a Baumslag-Solitar group (for those, see [16] and [26]). We use only one direct consequence of the main result of [26] which concerns injective homomorphisms.

Our main results are:

Theorem For any injective endomorphism of $B(m,n)$ where $|n| \neq |m|$ and $|nm| \neq 0$, the Reidemeister number is infinite. For any automorphism of $B(m,n)$ where $0 < |m| = |n|$ and $mn \neq 1$, the Reidemeister number is also infinite.

This result summarizes the results of Theorems 3.4, 4.4, , 5.4, 6.4 and Proposition 5.1 for the various values of m and n.

Theorem 7.1 The coincidence Reidemeister number is infinite for any pair of injective endomorphisms of the group $B(m,n)$, where $|n| \neq |m|$ and $|nm| \neq 0$.

We do not know if Theorems 5.4 and 6.4 are also true for injective homomorphisms. See Remarks 5.5 and 6.5.

We say that a group G has *property* R_∞ if any of its automorphisms ϕ has $R(\phi) = \infty$. After the preprint(in arXiv: math.GR-0405590) of this article was circulate, it was proved that the following groups have *property* R_∞ : (1) generalized Baumslag-Solitar groups, that is, finitely generated groups which act on a tree with all edge and vertex stabilizers infinite cyclic [27]; (2) lamplighter groups $Z_n \wr Z$ iff $2|n$ or $3|n$ [20]; (3) the solvable generalization Γ of $BS(1,n)$ given by the short exact sequence $1 \to Z[\frac{1}{n}] \to \Gamma \to Z^k \to 1$ as well as any group quasi-isometric to Γ [31]; (4) groups which are quasi-isometric to the Generalized Baumslag-Solitar groups [32]

(while this property is not a quasi-isometry invariant); (5) saturated weakly branch groups(including the Grigorchuk group and the Gupta-Sidki group) [15]; (6) the R. Thompson's groups [2]; (7) some finitely generated nilpotent groups of arbitrary Hirsch length [21].

We would like to complete the introduction with the following conjecture.

Conjecture 1.1. Any relatively hyperbolic group has *property* R_∞. In particular, any Kleinian group has *property* R_∞.

This paper is organized into six sections besides this one. In section 2, we make some simple reduction of the problem to certain cases and develop some preliminaries about the Reidemeister classes of a pair of homomorphisms between short exact sequences. In section 3, we study the case $B(\pm1, n)$ for $|n| > 1$, with main result Theorem 3.4. In section 4, we consider the cases $B(m, n)$ for $1 < |m| \neq |n| > 1$, with main result Theorem 4.4. In section 5, we consider the cases $B(m, -m)$ for $|m| > 0$, with main results Propositon 5.1 and Theorem 5.4. In section 6 we consider the cases $B(m, m)$ for $|m| > 1$, with main result Theorem 6.4. In section 7 we derive few results about the coincidence Reidemeister number, with main result Theorem 7.1

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This article is dedicated to the memory of Sasha Reznikov.

2. Generalities and Preliminaries

In this section we first describe few elementary properties of the groups $B(m, n)$ in order to reduce our problem to certain cases. Then we recall some facts about the Reidemeister classes of a pair of homomorphisms of a short exact sequence. Recall that *a group G has* property R_∞ *if any of its automorphisms ϕ has $R(\phi) = \infty$.*

Recall that the Baumslag-Solitar groups are indexed by pairs of integer numbers different from zero and they have the following presentation:

$$B(m, n) = \langle a, b : a^{-1}b^m a = b^n\rangle, m, n \neq 0.$$

The first observation is that for $m = n = 1$ this group is $\mathbb{Z} + \mathbb{Z}$. It is well known that this group does not have exponential growth and there are automorphisms $\phi : \mathbb{Z} + \mathbb{Z} \to \mathbb{Z} + \mathbb{Z}$ with finite Reidemeister number. So $\mathbb{Z} + \mathbb{Z}$ does not have property R_∞.

The second observation is that $B(m, n)$ is isomorphic to $B(-m, -n)$. It suffices to see that the relations $a^{-1}b^m a = b^n$ and $a^{-1}b^{-m}a = b^{-n}$ each one generates the same normal subgroup, since one relation is the inverse of the other.

The last observation is that $B(m, n)$ is isomorphic to $B(n, m)$. Suppose that $B(m, n) = \langle a, b : a^{-1}b^m a = b^n\rangle, m, n \neq 0$ and $B(n, m) = \langle c, d : c^{-1}d^n c = d^m\rangle, m, n \neq 0$. The map which sends $a \to c^{-1}$ and $b \to d$ extends to an isomorphism of the two groups.

Based on the above, we will consider only the groups $B(r, s),\ rs \neq 0, 1$ and we can show

Proposition 2.0 Each group $B(r, s),\ rs \neq 0, 1$ is isomorphic to some $B(m, n)$, where m, n satisfy $0 < m \leq |n|$ and $n \neq 1$.

So we will divide the problem into 4 cases. Case 1) is when $1 = m < |n|$; Case 2) is when $1 < m < |n|$; Case 3) when $0 < m = -n$; Case 4) when $1 < m = n$

The set of the Reidemeister classes of a pair of homomorphisms will be denoted by $R[\ ,\ ]$ and the number of such classes by $R(\ ,\ )$. When the two sequences are the same and one of the homomorphisms is the identity, then

we have the usual Reidemeister classes and Reidemeister number. The main reference for this section is [17] and more details can be found there.

Let us consider a diagram of two short exact sequences of groups and maps between these two sequences:

$$\begin{array}{ccccccccc} 1 & \to & H_1 & \overset{i_1}{\to} & G_1 & \overset{p_1}{\to} & Q_1 & \to & 1 \\ & & f' \downarrow\downarrow g' & & f \downarrow\downarrow g & & \overline{f} \downarrow\downarrow \overline{g} & & \\ 1 & \to & H_2 & \overset{i_2}{\to} & G_2 & \overset{p_2}{\to} & Q_2 & \to & 1 \end{array} \tag{2.1}$$

where $f' = f|_{H_1}$, $g' = g|_{H_1}$.

We recall that the set of the Reidemeister classes $R[f_1, f_2]$ relative to homomorphisms $f_1, f_2 : K \to \pi$ is the set of the equivalence classes of elements of π given by the following relation: $\alpha \sim f_2(\tau)\alpha f_1(\tau)^{-1}$ for $\alpha \in \pi$ and $\tau \in K$.

The diagram (2.1) provides maps between sets

$$R[f', g'] \overset{\widehat{i_2}}{\to} R[f, g] \overset{\widehat{p_2}}{\to} R[\overline{f}, \overline{g}]$$

where the last map is clearly surjective. An obvious consequence of this fact will be used to solve some of the cases that we will discuss, and that will appear below as Corollary 2.2. For the remaining cases we need further information about the above sequence and we will use Corollary 2.4.

Proposition 1.2 in [17] says

Proposition 2.1 Given the diagram (2.1) we have a short sequence of sets

$$R[f', g'] \overset{\widehat{i_2}}{\to} R[f, g] \overset{\widehat{p_2}}{\to} R[\overline{f}, \overline{g}]$$

where $\widehat{p_2}$ is surjective and $\widehat{p_2}^{-1}[1] = \mathrm{im}\,(\widehat{i_2})$, where 1 is the identity element of Q_2.

An immediate consequence is

Corollary 2.2 If $R(\overline{f}, \overline{g})$ is infinite, then $R(f, g)$ is also infinite.

Proof. Since $\widehat{p_2}$ is surjective the result follows. □

In order to study the injectivity of the map $\widehat{i_2}$, for each element $\overline{\alpha} \in Q_2$ let $H_2(\overline{\alpha}) = p_2^{-1}(\overline{\alpha})$, $C_{\overline{\alpha}} = \{\overline{\tau} \in Q_1 | \overline{g}(\overline{\tau})\overline{\alpha}\overline{f}(\overline{\tau}^{-1}) = \overline{\alpha}\}$ and let $R_{\overline{\alpha}}[f', g']$ be the set of equivalence classes of elements of $H_2(\overline{\alpha})$ given by the equivalence relation $\beta \sim g(\tau)\beta f(\tau^{-1})$, where $\beta \in H_2(\overline{\alpha})$ and $\tau \in p_1^{-1}(C_{\overline{\alpha}})$. Finally, let

$R[f_{\overline{\alpha}}, g_{\overline{\alpha}}]$ be the set of equivalence classes of elements of $H_2(\overline{\alpha})$ given by the relation $\beta \sim g(\tau)\beta f(\tau^{-1})$, where $\beta \in H_2(\overline{\alpha})$ and $\tau \in G_1$.

Proposition 1.2 in [17] says

Proposition 2.3 Two classes of $R(f_{\overline{\alpha}}, g_{\overline{\alpha}})$ represent the same class of $R(f, g)$ if and only if they belong to the same orbit by the action of $C_{\overline{\alpha}}$. Further the isotropy subgroup of this action at an element $[\beta]$ is $G_{[\beta]} = p_1(C_\beta) \subset C_{\overline{\alpha}}$ where $\beta \in [\beta]$.

An immediate consequence of this result is

Corollary 2.4 If $C_{\overline{\alpha}}$ is finite and $R(f_{\overline{\alpha}}, g_{\overline{\alpha}})$ is infinite for some α, then $R(f, g)$ is also infinite. In particular, this is the case if Q_2 is finite.

Proof. The orbits of the action of $C_{\overline{\alpha}}$ on $R[f_{\overline{\alpha}}, g_{\overline{\alpha}}]$ are finite. So we have an infinite number of orbits. The last part is an easy consequence of the first part. □

3. THE CASES $B(m, n)$, $1 = |m| < |n|$

From section 2 the cases in this section reduce to Case 1), namely $B(1, n)$ for $1 < |n|$. Let $|n| > 1$ and $B(1, n) = \langle a, b : a^{-1}ba = b^n, n > 1\rangle$.
Recall from [5] that the Baumslag-Solitar groups $B(1, n)$ are finitely generated solvable groups which are not virtually nilpotent. These groups have exponential growth [24], and they are not Gromov hyperbolic. Furthermore, those groups are metabelian and torsion free.

Consider the homomorphisms $|\ \ |_a : B(1, n) \longrightarrow \mathbb{Z}$ which associates for each word $w \in B(1, n)$ the sum of the exponents of a in the word. It is easy to see that this is a well defined map into $\mathbb{Z}$ which is surjective.

Proposition 3.1 We have a short exact sequence

$$0 \longrightarrow K \longrightarrow B(1, n) \xrightarrow{|\ \ |_a} \mathbb{Z} \longrightarrow 1,$$

where K, the kernel of the map $|\ \ |_a$, is the set of the elements which have the sum of the powers of a equal to zero. Furthermore, $B(1, n) = K \rtimes \mathbb{Z}$ (semi-direct product).

Proof. The first part is clear. The second part follows because $\mathbb{Z}$ is free, so the sequence splits. □

Proposition 3.2 The kernel K coincides with $N\langle b\rangle$, the normalizer of $\langle b\rangle$ in $B(1,n)$.

Proof. We have $N\langle b\rangle \subset K$. But the quotient $B/N\langle b\rangle$ has the following presentation: $\bar{a}^{-1}\bar{b}\bar{a} = \bar{b}^n, \bar{b} = 1$. Therefore this group is isomorphic to $\mathbb{Z}$ and the natural projection coincides with the map $|\ \ |_a$ under the obvious identification of $\mathbb{Z}$ with $B/N\langle b\rangle$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 \to N\langle b\rangle \to & B(1,n) \to & B/N\langle b\rangle \to & 1 \\ \downarrow & \downarrow & \downarrow & \\ 0 \to K \to & B(1,n) \to & \mathbb{Z} \to & 1 \end{array}$$

where the last vertical map is an isomorphism. From the well-known five Lemma the result follows. □

The groups $B(1,n)$ are metabelian. Let ϵ be the sign of n. We recall the result that $B(1,n)$ is isomorphic to $\mathbb{Z}[1/|n|] \rtimes_\theta \mathbb{Z}$ where the action of $\mathbb{Z}$ on $\mathbb{Z}[1/|n|]$ is given by $\theta(1)(x) = x/n^\epsilon$. To see this, first observe that the map defined by $\phi(a) = (0,1)$ and $\phi(b) = (1,0)$ extends to a unique homomorphism $\phi : B \to \mathbb{Z}[1/|n|] \rtimes \mathbb{Z}$ which is clearly surjective. It suffices to show that this homomorphism is injective. Consider a word $w = a^{r_1}b^{s_1}...a^{r_t}b^{s_t}$ such that $r_1 + ... + r_t = 0$. Thus $w \in K$, and using the relation of the group this word is equivalent to $b^{s_1/n^{\epsilon r_1}} b^{s_2/n^{\epsilon(r_1+r_2)}} .... b^{s_{t-1}/n^{\epsilon(r_1+...+r_{t-1})}} .b^{s_t}$. If we apply ϕ to this element, which belongs to the kernel of ϕ, we obtain that the sum of the powers $s_1/n^{r_1} + s_2/n^{\epsilon(r_1+r_2)} + ... + s_{t-1}/n^{\epsilon(r_1+...+r_{t-1})} + s_t$ is zero. But this means that w is the trivial element, hence ϕ restricted to K is injective. Therefore the result follows.

Proposition 3.3 Any homomorphism $\phi : B(1,n) \to B(1,n)$ is a homomorphism of the short exact sequence given in Proposition 3.2.

Proof. Let $\bar{\phi}$ be the homomorphism induced by ϕ on the abelianization of $B(1,n)$. The abelianization of $B(1,n)$, denoted by $B(1,n)_{ab}$, is isomorphic to $\mathbb{Z}_{|n-1|} + \mathbb{Z}$. The torsion elements of $B(1,n)_{ab}$ form a subgroup isomorphic to $\mathbb{Z}_{|n-1|}$ which is invariant under any homomorphism. The preimage of this

subgroup under the projection to the abelianization $B(1,n) \to B(1,n)_{ab}$ is exactly the subgroup $N(b)$, i.e., the elements represented by words where the sum of the powers of a is zero. So it follows that $N(b)$ is mapped into $N(b)$. $\square$

Theorem 3.4 For any injective homomorphism of $B(1,n)$ the Reidemeister number is infinite.

Proof. Let ϕ be an injective endomorphism. By Proposition 3.3 it is an endomorphism of the short exact sequence given by Proposition 3.2. The induced homomorphism on the quotient is a non-trivial endomorphism of $\mathbb{Z}$. Otherwise we would have an injective homomorphism from the non-abelian group $B(1,n)$ into the abelian group K. If the induced endomorphism $\bar{\phi}$ is the identity, by Corollary 2.2 the number of Reidemeister classes is infinite and the result follows. So, assume that $\bar{\phi}$ is multiplication by $k \neq 0, 1$ and we will get a contradiction. Now we claim that there is no injective endomorphism of $B(1,n)$ such that the induced homomorphism on the quotient is multiplication by k with $k \neq 0, 1$. When we apply the homomorphism ϕ to the relation $a^{-1}ba = b^n$, using the isomorphism $B(1,n) \to \mathbb{Z}[1/n] \rtimes \mathbb{Z}$ we obtain: $a^{-k}\phi(b)a^k = (n^k\phi(b), 0) = (n\phi(b), 0)$, which implies that either $n^{1-k} = 1$ or $\phi(b) = 0$. Since $\phi(b) \neq 0$ and n is neither 1 or -1 we get a contradiction and the result follows. $\square$

Remark 3.5 From the proof above we conclude that any injective homomorphism $\varphi : B(1,n) \to B(1,n)$ has the property that it induces the identity on the quotient $\mathbb{Z}$ given by the short exact sequence in Propositon 3.2. This fact will be used to study coincidence Reidemeister classes in section 7.

4. THE CASE $B(m,n)$, $1 < |m| \neq |n| > 1$

From section 2 the cases in this section reduce to Case 2), namely (m,n) for $1 < m < |n|$. The groups in this section are more complicated than the ones in the previous section. Nevertheless in order to obtain the results we will use a similar procedure to the one in the previous section. Let

$1 < m < |n|$ and $B(m,n) = \langle a, b : a^{-1}b^m a = b^n \rangle$. Recall that such groups are non-virtually solvable.

Consider the homomorphism $|\ \ |_a : B(m,n) \longrightarrow \mathbb{Z}$ which associates to each word $w \in B(m,n)$ the sum of the powers of a in the word. It is easy to see that this is a well defined homomorphism into $\mathbb{Z}$ which is surjective.

Proposition 4.1 We have a short exact sequence

$$0 \longrightarrow K \longrightarrow B(m,n) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where K, the kernel of the map $|\ \ |_a$, is the set of the elements which have the sum of the powers of a equals to zero. Furthermore, $B(m,n) = K \rtimes \mathbb{Z}$ is a semi-direct product where the action is given with respect to some fixed section.

Proof. The first part is clear. The second part follows because $\mathbb{Z}$ is free, so the sequence splits. Since the kernel K is not abelian, the action is defined with respect to a specific section (see [3]). $\square$

Proposition 4.2 The kernel K coincides with $N\langle b \rangle$ which is the normalizer of $\langle b \rangle$ in $B(m,n)$.

Proof. Similar to Proposition 3.2. $\square$

Proposition 4.3 Any homomorphism $\phi : B(m,n) \to B(m,n)$ is a homomorphism of the short exact sequence given in Proposition 4.1.

Proof. Let $\bar{\phi}$ be the homomorphism induced by ϕ on the abelianized of $B(m,n)$. The abelianized of $B(m,n)$, denoted by $B(m,n)_{ab}$, is isomorphic to $\mathbb{Z}_{|n-m|} + \mathbb{Z}$. The torsion elements of $B(m,n)_{ab}$ form a subgroup isomorphic to $\mathbb{Z}_{|n-m|}$ which is invariant under any homomorphism. The preimage of this subgroup under the projection to the abelianized $B(m,n) \to B(m,n)_{ab}$ is exactly the subgroup $N(b)$, i.e., the elements represented by words where the sum of the powers of a is zero. So it follows that $N(b)$ is mapped into $N(b)$. $\square$

In order to have a homomorphism ϕ of $B(m,n)$ which has finite Reidemeister number, the induced map on the quotient $\mathbb{Z}$ must be different from the identity by the same argument used in the proof of Theorem 3.4.

Now we will give a presentation of the group K. The group K is generated by the elements $g_i = a^{-i}ba^i$ $i \in \mathbb{Z}$ which satisfy the following relations: $\{1 = a^{-j}(a^{-1}b^m ab^{-n})a^j = g_{j+1}^m g_j^{-n}\}$ for all integers j. This presentation is a consequence of the Bass-Serre theory, see [4], Theorem 27, page 211.

Now we will prove the main result of this section. Denote by K_{ab} the abelianization of K.

Theorem 4.4 For any injective homomorphism of $B(m,n)$ the Reidemeister number is infinite.

Proof. Let us consider the short exact sequence, obtained from the short exact sequence given in Proposition 4.1, by taking the quotient with the commutators subgroup of K, i.e.

$$0 \longrightarrow K_{ab} \longrightarrow B(m,n)/[K,K] \longrightarrow \mathbb{Z} \longrightarrow 1.$$

Thus we obtain a short exact sequence where the kernel K_{ab} is abelian. From the presentation of K we obtain a presentation of K_{ab} given as follows: It is generated by the elements g_i, $i \in Z$, which satisfy the following relations: $\{1 = g_{j+1}^m g_j^{-n}, g_i g_j = g_j g_i\}$ for all integers i, j. This presentation is the same as the quotient of the free abelian group generated by the elements g_i, $i \in \mathbb{Z}$ (so the direct sum of $\mathbb{Z}'s$ indexed by $\mathbb{Z}$), modulo the subgroup generated by the relations $\{1 = g_{j+1}^m g_j^{-n}\}$. Thus an element can be regarded as an equivalence class of a sequence of integers indexed by $\mathbb{Z}$, where the elements of the sequence are zero but a finite number. By abuse of notation we also denote by ϕ the induced endomorphism on $B(m,n)/[K,K]$.

Let $\phi(a) = a^k\theta$ for $\theta \in K_{ab}$ and $k \neq 1$. Recall that if $k = 1$ it follows immediately that the Reidemeister number is infinite. Since the kernel of the extension is abelian, after applying ϕ to the relation $a^{-1}b^m a = b^n$ we obtain

$$\theta^{-1}a^{-k}\phi((b)^m)a^k\theta = a^{-k}\phi(b)^m a^k = \phi(b^n) = \phi(b)^n.$$

From the main result of [26], the element $\phi(b)$ is a conjugate of a power of b, i.e., $\phi(b) = \gamma b^r \gamma^{-1}$ for some $r \neq 0$. In the abelianization the element $\gamma b^r \gamma^{-1}$ is the same as the element $a^s ba^{-s}$ for some integer s. So any power of $\phi(b)$ with exponent different from zero is a nontrivial element. Now we take both sides of the equation above to the power m^k. If $k = 0$ it follows immediately

that $m = n$. Let us take $k > 1$. After applying the relation several times we obtain

$$a^{-k}\phi(b)^{mm^k}a^k = \phi(b)^{mn^k} = \phi(b)^{nm^k}.$$

Therefore it follows that $mn^k = nmk$ or $n^{k-1} = m^{k-1}$. If n is positive, since $k \neq 1$, then the only solutions are $m = n$, which is a contradiction. If n is negative then the only solutions are, $m = n$ or $m = -n$ and k even. In either case we get a contradiction. The case where $k < 0$ is similar and the result follows. □

Remark 4.5 The Proposition 4.1 certainly holds for $n = -m$. If we apply the proof of the Theorem 4.4 above for the group $B(m, -m)$ we can conclude that any injective homomorphism $\varphi : B(m, -m) \to B(m, -m)$ has the property that it induces the multiplication by an odd number on the quotient $\mathbb{Z}$, where $\mathbb{Z}$ is given by the short exact sequence used in the proof of the Theorem 4.4.

5. THE CASE $B(m, -m)$, $1 \leq |m|$

From section 2 the cases in this section reduce to Case 3), namely $B(m, -m)$ for $0 < m$.

We will start with the group $B(1, -1)$. In this case it has been proved in (see [21]) that this group has the R_∞ property. For sake of completeness we write another proof (which is also known to the second author of [21]) where the techniques is more close to the ones used in this work. The group $B(1, -1)$ is isomorphic to the fundamental group of the Klein bottle.

Proposition 5.1 For any automorphism ϕ of $\mathbb{Z} \rtimes \mathbb{Z}$ the Reidemeister number is infinite.

Proof. Let us consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \rtimes \mathbb{Z} \to \mathbb{Z},$$

where the inclusion $\mathbb{Z} \to \mathbb{Z} \rtimes \mathbb{Z}$ sends $1 \to x$. It is well know that $\mathbb{Z}$ is characteristic in $\mathbb{Z} \rtimes \mathbb{Z}$, so any homomorphism $\varphi : \mathbb{Z} \rtimes \mathbb{Z} \to \mathbb{Z} \rtimes \mathbb{Z}$ induces a homomorphism of short exact sequence. Let φ be an automorphism. Then

the induced automorhism on the quotient $\bar{\varphi} : \mathbb{Z} \to \mathbb{Z}$ is either the identity or minus the identity. In the first case we have that the Reidemeister number of φ is infinite and the result follows. So let us assume that $\bar{\varphi}$ is $-id$. The induced map on the fiber φ' is also either the identity or minus the identity. In either case, in order to compute the Reidemeister number of φ, by means of the formula given in [18], Lemma 2.1 we need to consider the homomorphism given by the composition of φ' with the conjugation by y, which is the multiplication by -1, i.e. $-\varphi'$. So either φ' or $-\varphi'$ is the identity. Again by the formula given in [18], Lemma 2.1, the result follows.

□

The above result is not true for injective homomorphisms. Take for example the homomorphism defined by $\varphi(x) = x^2, \varphi(y) = y^3$. It is an injective homomorphism and $R(\varphi)$ is 4.

From now on let $1 < m$. The groups $B(m, -m)$, in contrast with others Baumslag-Solitar groups already considered, have subgroups isomorphic to $\mathbb{Z} \rtimes \mathbb{Z} = B(1, -1)$, the fundamental group of the Klein bottle.

It is straightforward to verify that Propositions 4.1, 4.2 and 4.3 are also true for $m = -n$ (this is not the case for Proposition 4.3 when $m = n$). So we have a short exact sequence

$$0 \longrightarrow K \longrightarrow B(m, -m) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where K is the kernel of the map $| \quad |_a$. This kernel coincides with the normal subgroup generated by the element b and any homomorphism $\phi : B(m, -m) \to B(m, -m)$ is a homomorphism of the short exact sequence. Denote by $\bar{\phi}$ the induced homomorphism on the quotient $\mathbb{Z}$ and by $\phi' : K \to K$ the restriction of ϕ. Our proof have some similarities with the proof for the group $B(1, -1)$.

Proposition 5.2 Given any automorphism $\phi : B(m, -m) \to B(m, -m)$, then the induced automorphism on the quotient $\bar{\phi}$ is either the identity or minus the identity. In the former case we have $R(\phi)$ infinite.

Proof. Follows immediatly from Corollary 2.2.

□

Proposition 5.3 Given any automorphism $\phi : B(m,-m) \to B(m,-m)$ such that the induced homomorphism on the quotient $\bar{\phi} : \mathbb{Z} \to \mathbb{Z}$ is multiplication by -1, either the automorphism ϕ' or the automorphism $\tau_a \circ \phi'$, where τ_a is the conjugation by the element a, have infinite Reidemeister number.

Proof. From [26] $\phi'(b) = a^i b^\epsilon a^{-i}$ for some integer i, where ϵ is either 1 or -1 since we have an automorphism. For the other automorphism we have $\tau_a \circ \phi'(b) = a^{i+1} b^\epsilon a^{-i-1}$. We certainly have either $\epsilon = (-1)^i$ or $\epsilon = (-1)^{i+1}$. Let φ be ϕ' or $\tau_a \circ \phi'$ according to $\epsilon = (-1)^i$ or $\epsilon = (-1)^{i+1}$, respectively. A presentation of the group K was given in section 4 before Theorem 4.4. Consider the extra relation in K given by $b = ab^{-1}a^{-1}$. Then it follows that the quotient group is $\mathbb{Z}$ and the automorphism φ induces a homomorphism on the quotient which agrees with the identity. So it follows that the Reidemeister number of φ is infinite. □

Theorem 5.4 For any automorphism of $B(m,-m)$ the Reidemeister number is infinite.

Proof. Let $\phi : B(m,-m) \to B(m,-m)$ an automorphism. From Proposition 5.2 we can assume that $\bar{\phi}$ is multiplication by -1. From Propositon 5.3 we know that either ϕ' or $\tau_a \circ \phi'$ has infinite Reidemeister number. By the formula given in [18], Lemma 2.1, the result follows. □

Remark 5.5 We do not know an example of an injective homomorphism on $B(m,-m)$, for $m > 1$, which has finite Reidemeister number.

6. THE CASE $B(m,m)$, $|m| > 1$

From section 2 the cases in this section reduce to Case 4), namely $B(m,m)$ for $1 < m$. The proof of this case is not similar to the previous cases.

As we noted before, if $m = 1$, the group is $\mathbb{Z} + \mathbb{Z}$. Then there are automorphisms of the group which have a finite number of Reidemeister classes. For $m > 1$, in order to study its automorphisms, we describe the groups

$B(m,m)$ as certain extensions. Finally we show that any automorphism of this group has infinite Reidemeister number.

These groups, in contrast with the Baumslag-Solitar groups already considered, have subgroups isomorphic to $\mathbb{Z}+\mathbb{Z}$. We remark that for $n = 2$ this is not the fundamental group of the Klein bottle. There is a surjection from $B(2,2)$ onto the fundamental group of the Klein bottle.

We start by describing these groups. Let $| \ \ |_b : B(m,m) \to \mathbb{Z}$ be the homomorphism which associates to a word the sum of the powers of b which appears in the word. This is a well defined surjective homomorphism and we have

Proposition 6.1 There is a splitting short exact sequence:

$$0 \to F \to B(m,m) \to \mathbb{Z} \to 1,$$

where F is the free group on m generators $x_1, ..., x_m$ and the last map is $| \ \ |_b$. Further, the action of the generator $1 \in Z$ is the automorphism of F which, for $j < m$, sends x_j to x_{j+1} and x_m to x_1.

Proof. Let $F \rtimes \mathbb{Z}$ be the semi-direct product of F by $\mathbb{Z}$, where F is the free group on $x_1, ..., x_m$ and the action is given by the automorphism of F which, for $j < m$, maps x_j to x_{j+1} and x_m to x_1. We will show that $B(m,m)$ is isomorphic to $F \rtimes \mathbb{Z}$. For this consider the map $\psi : \{a, b\} \to F \rtimes \mathbb{Z}$ which sends a to x_1 and b to $1 \in Z$. This map extends to a homomorphism $B(m,m) \to F \rtimes Z$, which we also denote by ψ, since the relation which defines the group $B(m,m)$ is preserved by the map. Also ψ is a homomorphism of short exact sequences. The map restricted to the kernel of $| \ \ |_b$ is surjective to the free group F. Also the kernel admits a set of generators with cardinality n. So the map restricted to the kernel is an isomorphism and the result follows.

$\square$

Proposition 6.1 above shows that the groups $B(m,m)$ are policyclic. For more about the Reidemeister number of these groups see [9].

Proposition 6.2 The center of $B(m,m)$ is the subgroup generated by b^m. Moreover, any injective homomorphism $\phi : B(m,m) \to B(m,m)$ leaves the center invariant.

Proof. For the first part, from Proposition 6.1, we know that $B(m,m)$ is of the form $F \rtimes \mathbb{Z}$. Let $(w, b^r) \in F \rtimes \mathbb{Z}$ be in the center and $(v, 1) \in F \rtimes \mathbb{Z}$ where v is an arbitrary element of F. We have $(w, b^r)(v, 1) = (w \cdot b^r(v), b^r)$ and $(v, 1)(w, b^r) = (v \cdot w, b^r)$. We can assume that w is a word written in the reduced form which starts with $x_i^{m_i}$, for some $1 \leq i \leq m$. Let r_0 be the integer, $0 \leq r_0 \leq m-1$, congruent to r mod m. Now we consider three cases:

(1) $r_0 = 0$. Take $v = x_{i+1}$ if $i < m$ and $v = x_1$ if $i = m$. We claim that $w.b^r(v) \neq v.w$, so the elements do not commute. To see that they do not commute observe first that $v.w$ is in the reduced form. If $w.b^r(v)$ is not reduced they cannot be equal. If it is reduced, also they can not be equal either, since they start with different letters. The argument above does not work if $w = 1$, but this is the case where the element is in the center.

(2) $r_0 \neq 0$ and $w \neq 1$. Take $v = x_i^{m_i}$. Again $v.w$ is in the reduced form starting with $x_i^{2m_i}$. If $w \cdot b^r(v)$ is not reduced they cannot be equal. If it is reduced, also they cannot be equal either, since they start with different powers of x_i, even if the word contains only one letter, since $b^r(v)$ is not a power of x_i (r is not congruent to 0 mod m).

(3) $r_0 \neq 0$ and $w = 1$. Then $r = km + r_0$, and from the relation in the group it follows that $a^{-1}b^r a = a^{-1}b^{km+r_0}ra = b^{km}a^{-1}b^{r_0}a$. But $a^{-1}b^{r_0}a = b^{r_0}$ implies $b^{r_0}ab^{-r_0} = a$, which in terms of the notation in Proposition 5.1 means $x_1 = x_{r_0}$, which is a contradiction. So the result follows.

For the second part we have to show that $\phi(b^m)$ is in the center. Since ϕ is injective, from the main result of [26], the element $\phi(b)$ is conjugated to a power of b, i.e., $\phi(b) = \gamma b^r \gamma^{-1}$ for some $r \neq 0$. Therefore $\phi(b^m) = \gamma(b^r)^m\gamma^{-1} = bmr$ and the result follows.

□

Next we consider the group which is the quotient of $F \rtimes \mathbb{Z}$ by the center, where the center is the subgroup $< b^m >$. This quotient is isomorphic to $F \rtimes \mathbb{Z}_n$ and we denote the image of the generator b in $\mathbb{Z}$ by $\bar{b}$ in $\mathbb{Z}_m$.

Proposition 6.3 Any automorphism of the group $F \rtimes \mathbb{Z}_m$ has infinite Reidemeister number.

Proof. We know that F is the free group on the letters $x_1, ..., x_m$ and let $\theta : F \rtimes \mathbb{Z} \to \mathbb{Z}_n$ be the homomorphism defined by $\theta(x_i) = 1$ and $\theta(\bar{b}) = 0$. The kernel of this homomorphism defines a subgroup of $F \rtimes \mathbb{Z}$ of index m which is isomorphic to $F' \rtimes \mathbb{Z}_n$, where F' is the kernel of the homomorphism θ restricted to F. Now we claim that F' is invariant with respect to any homomorphism, i.e, F' is characteristic. Let $(w, \bar{1})$ be an arbitrary element of the subgroup F' with $w \neq 1$. First observe that $\theta(\phi(x_i)) = \theta(\phi(x_1))$, for all i. This follows by induction since $x_{i+1} = \bar{b}.x_i.\bar{b}^{-1}$, $\theta(\phi(x_{i+1})) = \theta(\phi(\bar{b})).\theta(\phi(x_i)).\theta(\phi(\bar{b}^{-1})) = \theta(\phi(x_i))$. Therefore $\theta(\phi(w, \bar{1})) = \theta((w, \bar{1}))\theta(\phi(x_1))$ and the subgroup F' is invariant. Therefore the automorphism ϕ provides an automorphism of the short exact sequence

$$0 \to F' \to F \rtimes \mathbb{Z}_m \to \mathbb{Z}_m + \mathbb{Z}_m \to 0$$

where the restriction to the kernel is an automorphism of a free group of finite rank. Hence, by the Corollary 2.4 the result follows. □

Now we proof the main result.

Theorem 6.4 Any automorphism ϕ of $B(m, m)$ has an infinite Reidemeister number.

Proof. Any automorphism ϕ, from Proposition 5.2, induces an automorphism on $F \rtimes \mathbb{Z}_m$, which we denote by $\bar{\phi}$. In order to prove that ϕ has an infinite Reidemeister number, it suffices to show the same statement for $\bar{\phi}$. By Proposition 5.3 the statement is true for $\bar{\phi}$, so the Theorem follows. □

Remark 6.5 Proposition 6.3 and Theorem 6.4 use only Proposition 6.2 for automorphisms, which, under this assumption, its second part of the Proposition 6.2 becomes obvious. Nevertheless, using Proposition 6.2 as stated, it is not difficult to see that Proposition 6.3 and Theorem 6.4 can be extended for injective homomorphisms if one knows the result for injective homomorphisms of a free group of finite rank. However this is still an open question.

7. Coincidence Reidemeister classes

For a pair of homomorphisms $\phi, \psi : G \to G$ one can ask similarly when a pair of homomorphisms (ϕ, ψ) has infinite coincidence Reidemeister number. If one of the homomorphisms, let us say ϕ, is an automorphism, then the

problem is equivalent to the classical problem for the homomorphism $\phi^{-1}\circ\psi$. So we can apply all the results above. There are many cases which can be obtained from the case of one homomorphism. Theorems 3.4, 4.4, 6.2, and 6.4 can be generalized to coincidence as follows:

Theorem 7.1 The coincidence Reidemeister number is infinite for any pair of injective endomorphisms of the group $B(m,n)$, where $|n| \neq |m|$ and $|nm| \neq 0$.

Proof. For the cases in question, we have proved that an injective homomorphism induces a homomorphism of the short exact sequences given by Propositions 3.1 and 4.1, depending on the values of m and n, respectively. Further in any of the cases above, by the proof of the Theorems 3.4, 4.4, 6.2 and 6.4, we have that the induced homomorphisms $\bar{\phi}$ and $\bar{\psi}$ on the quotients are the identity on $\mathbb{Z}$. So the pair $(\bar{\phi},\bar{\psi})$ has infinite coincidence Reidemeister number and the result follows from Corollary 2.2. □

The extension of Theorem 7.1 for the groups $B(m,m)$, will follow if the same result is true for a pair of injective homomophisms of a free group of finite rank. But, as pointed out in Remark 6.5, this is not known even if one of the homomorphisms is the identity.

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